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ANALYTICAL SOLUTION OF THE RADIATION TRANSFER EQUATION FOR A THREE-DIMENSIONAL VOLUME OF A DISPERSE MEDIUM WITH AN ARBITRARY SCATTERING INDICATRIX

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An analytical solution of the boundary-value problem for the equation of radiation transfer in a three-dimensional disperse medium with an arbitrary scattering indicatrix has been found. The solution has the form of an expansion in the basis of the finite-dimensional functional space of the author's special G functions that in full measure take into account information on the angular structure of a real scattering indicatrix.

Keywords: *radiation transfer equation, multiple scattering of light, method of G functions, Green's function of transport theory, analytical solution of radiation transfer equation.*

Introduction. Since the time the radiation transfer equation (RTE) was obtained, attempts have been undertaken repeatedly to solve it analytically for an arbitrary scattering indicatrix [1]. They were stimulated by the necessity of solving practically important problems in various fields of astrophysics, gas dynamics, nuclear power engineering, atmospheric optics, hydro-optics of disperse media, etc. The main difficulty considered to be unsurmountable in the attempts at analytical solution of the RTE is, as is known, associated with the necessity of taking into account an infinitely large number of terms in expansions of the scattering indicatrix in spherical functions [2]. The desire to overcome this obstacle has led to the development of numerical [3] and approximate analytical methods such as, e.g., diffusion and small-angle diffusion approximations [4] and the P_n approximation of the method of spherical harmonics [5] that uses only several first terms of the series to represent an indicatrix. Note that for a satisfactory approximation of, say, the aerosol scattering indicatrix no less than four hundred terms of the series are needed [6]. Therefore approximate solutions are highly idealized and in many cases do not allow one to obtain physically correct results. Thus, for example, the method of spherical harmonics leads to errors in the region of small angles of scattering. On the contrary, the method of small-angle approximation ignores very important information in the region of large scattering angles. In order to solve the RTE, the method of the author special G functions was implemented in the present work; the method is free of the indicated drawbacks. It presupposes the abandonment of the infinite systems of spherical functions [7] for representing the solution and the construction of a new system of special G functions that form a finite-dimensional functional space.

Formulation of the Boundary-Value Problem for the Radiation Transfer Equation. Let a monochromatic monodirected beam of light impinge normally onto a three-dimensional volume of a disperse medium with an arbitrary scattering indicatrix. We will select the volume of the medium in the form of a rectangular parallelepiped oriented along the Cartesian coordinate system axes $0xyz$. The surface Γ of the medium is determined by six faces with the coordinates: $x_\Gamma = 0, a$; $y_\Gamma = 0, b$; $z_\Gamma = 0, c$ and with the corresponding orientations $\mathbf{n}$ of their internal normals: $n_x = +1, -1$; $n_y = +1, -1$; $n_z = +1, -1$. The volume of the medium is $V = a \times b \times c$.

The integrodifferential transfer equation [2]

$$(\mathbf{\Omega}\nabla)\, I(\mathbf{r}, \mathbf{\Omega}) + \varepsilon I(\mathbf{r}, \mathbf{\Omega}) = \frac{\varepsilon\Lambda}{4\pi}\int\limits_{\mathbf{\Omega}'} x(\mathbf{\Omega}\cdot\mathbf{\Omega}')\, I(\mathbf{r}, \mathbf{\Omega})\, d\Omega' + B_1(\mathbf{r}, \mathbf{\Omega}) , \tag{1}$$

supplemented by the boundary conditions

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$$I(\mathbf{r}, \mathbf{\Omega}) = 0\,, \quad (\mathbf{n}\cdot\mathbf{\Omega}) > 0\,, \quad \mathbf{r} = \mathbf{r}_\Gamma\,, \quad \mathbf{n} = \mathbf{n}_\Gamma\,, \tag{2}$$

determines the boundary-value problem of the theory of radiation transfer in a disperse medium under the conditions of multiple scattering. Here $I(\mathbf{r}, \mathbf{\Omega})$ is the diffuse radiation intensity as a function of the space coordinates $\mathbf{r} = \mathbf{r}(x, y, z)$ and of the direction of sighting $\mathbf{\Omega} = \mathbf{\Omega}(\vartheta, \varphi)$; $x(\mathbf{\Omega}\cdot\mathbf{\Omega}') = 4\pi f\,(\mathbf{\Omega}\cdot\mathbf{\Omega}')$ is the light scattering indicatrix (phase function), $(\mathbf{\Omega}\cdot\mathbf{\Omega}') = \cos\theta = \mu_\theta$ (in what follows the subscript at μ will be omitted); $\Lambda = \sigma/\varepsilon$ is the probability of survival of the light quantum (single scattering albedo); ε and σ are the volume index of attenuation and scattering of light, respectively; $B_1(\mathbf{r}, \mathbf{\Omega}) = (1/4\pi)\sigma x(\mathbf{\Omega}\cdot\mathbf{\Omega}_0)\pi F_0 \exp(-\varepsilon z)$ is the function of the sources of single scattering. In solving the problem, the scattering indicatrix and the function of sources are assumed to be given.

System of Orthogonal *G* Functions of a Special Functional Space. A qualitative analysis of the RTE shows that its properties differ substantially from the properties of the classical equations of mathematical physics. The method of spherical harmonics that was well developed for their solution does not yield the desired results in the case of RTE. In order to solve the problems of the theory of radiation transfer it is applied only in the simplest cases and is related to the class of approximate methods. From the analysis of integrodifferential equation (1) containing a differential operator in the form of a derivative with respect to direction and an integral operator in the form of a collision integral, it follows that it is necessary that for each of them different orthogonal functions be used which, however, could satisfy the characteristic properties of the RTE symmetry. To construct such functions as an initial system we will select a system of independent functions $\{U_j(\mu),\ j = 0, 1\}$ that are genetically related to the RTE. The condition of this relationship is the determining one for constructing a system of G functions. By linear transformation we will find auxiliary functions $\Psi_l(\mu)$, $l = 0, 1, 2$ that can be conveniently written in the form of matrix representation:

$$\begin{bmatrix} \Psi_0(\mu) \\ \Psi_1(\mu) \\ \Psi_2(\mu) \end{bmatrix} = \begin{bmatrix} U_0(\mu) \\ U_1(\mu) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_{10} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ g_0(\mu) \\ g_1(\mu) \end{bmatrix}. \tag{3}$$

The one-dimensional functions $g_l(\mu)$, $l \le N = 1$, in Eq. (3) are determined by normalizing $g_l(\mu) = \Psi_l(\mu)(N_l)^{-1}$ beginning from $g_0(\mu) = U_0(\mu)(N_0)^{-1}$. The square of the normalizing factor N_i is calculated by means of the equation

$$N_l^2 = \frac{2l+1}{2} \int_{-1}^{1} [\Psi_l(\mu)]^2\, d\mu\,. \tag{4}$$

It is obvious that for the values of the index $2 \le l < \infty$ all the functions $\Psi_l(\mu)$ and $g_l(\mu)$ are equal to zero. The value of the matrix element a_{10} in (3) is determined based on the requirement of the orthogonality of the auxiliary function $\Psi_1(\mu) = U_1(\mu) + a_{10}g_0(\mu)$ relative to the function $g_0(\mu)$ in the interval $\mu \in [-1, +1]$ according to the equation

$$a_{10} = -\frac{1}{2} \int_{-1}^{+1} U_1(\mu)\, g_0(\mu)\, d\mu\,. \tag{5}$$

The orthogonal functions $g_l(\mu)$ obtained in this way satisfy the normalization condition:

$$\int_{-1}^{1} g_l(\mu)\, g_{l'}(\mu)\, d\mu = \delta_{ll'} \frac{2}{2l+1}\,, \tag{6}$$

where $\delta_{ll'}$ is the Kronecker symbol. The set of functions $\{g_l(\mu)\}$ will be supplemented by a system of associated functions $\{g_l^m(\mu)\}$ determined by differentiation of $g_l(\mu)$:

$$g_l^m(\mu) = (1-\mu^2)^{m/2} \frac{d^m}{d\mu^m} g_l(\mu)\,. \tag{7}$$

On a single sphere that corresponds to a set of the directions of a light beam $\mathbf{\Omega} \in \{[-1, +1] \times [0, 2\pi]\}$, where $\mu = \cos\vartheta \in [-1, +1]$, $\varphi \in [0, 2\pi]$, we introduce two-dimensional orthogonal functions $G_l^m(\mu, \varphi)$ with the aid of the following expressions:

$$G_l^m(\mathbf{\Omega}) = A_l B_{Gl}^m C^m g_l^m(\mu) \exp(im\varphi), \quad l = 0, 1; \quad -l \le m \le l,$$

$$A_l = \left[\frac{2l+1}{4\pi}\right]^{1/2}, \quad B_{Gl}^m = (A_l)^{-1} \int_{-1}^{1}\int_{0}^{2\pi} (1-\mu^2)^m \left[\frac{d^m}{d\mu^m} g_l(\mu)\right]^2 d\mu d\varphi, \quad C^m = (-1)^{(m+|m|)/2}. \tag{8}$$

The functions $G_l^m(\mu, \varphi)$ constructed in this way are orthonormalized according to the condition

$$\iint_{\Omega} G_l^{*m}(\mu, \varphi)\, G_{l'}^{m'}(\mu, \varphi)\, d\mu d\varphi = \delta_{ll'}^{mm'}, \tag{9}$$

where $\delta_{ll'}^{mm'}$ is the Kronecker symbol over the indices l and m. They satisfy the theorem of summation on a sphere:

$$g_l(\mathbf{\Omega}\cdot\mathbf{\Omega}') = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} G_l^{*m}(\mathbf{\Omega})\, G_l^m(\mathbf{\Omega}'). \tag{10}$$

Such is the general scheme of the construction of a set of orthogonal special G functions: $\{g_l(\mu), g_l^m(\mu), G_l^m(\mu, \varphi); l \le N = 1; -l \le m \le l\}$. The explicit form of the mathematical expressions of G functions can be found provided account is taken of the condition of precise representation of the real light scattering indicatrix by the sum

$$x(\mu) = \sum_{l=0}^{N} x_l g_l(\mu) = \sum_{l=0}^{N} \sum_{m=-l}^{l} [A_l]^{-2} x_l G_l^{*m}(\mathbf{\Omega})\, G_l^m(\mathbf{\Omega}'), \tag{11}$$

where x_l are the coefficients of expansion of the scattering indicatrix in G functions:

$$x_l = \frac{2l+1}{2} \int_{-1}^{1} x(\mu)\, g_l(\mu)\, d\mu. \tag{12}$$

Having substituted the coefficients (12) into (11), it can be easily seen that the expansion of the indicatrix in the G functions satisfies equality (11) and the Parseval–Steklov condition of closeness:

$$\| x(\mu) \|^2 = \sum_{l=0}^{1} \frac{2}{2l+1} x_l^2, \tag{13}$$

where the designation of the squared norm of the indicatrix [8] is used:

$$\| x(\mu) \|^2 = \int_{-1}^{1} [x(\mu)]^2\, d\mu. \tag{14}$$

It is evident that a system of special G functions can be constructed in this way for an arbitrary real light scattering indicatrix $x(\mu)$. For this purpose, in the above-given scheme we should take $U_0(\mu) = 1$, $U_1(\mu) = x(\mu)$ as initial functions $\{U_j(\mu)\}$. Direct calculations for $U_j(\mu)$ performed for a complete set of possible values of indices l and m lead to explicit expressions of the unknown orthonormalized functions $G_l^m(\mu, \varphi)$:

$$G_0^0(\mu,\varphi)=G_0(\mu,\varphi)=\left[\frac{1}{4\pi}\right]^{1/2} g_0^0(\mu)\,,\ \ g_0^0(\mu)=g_0(\mu)=1\,,$$

$$G_1^0(\mu,\varphi)=G_1(\mu,\varphi)=\left[\frac{3}{4\pi}\right]^{1/2} g_1^0(\mu)\,,\ \ g_1^0(\mu)=g_1(\mu)=\left\{\frac{3}{2}\left[\,\|x(\mu)\|^2-2\right]\right\}^{-1/2}[x(\mu)-1]\,,$$

$$G_1^1(\mu,\varphi)=(-1)\left[\frac{3}{4\pi}\right]^{1/2} B_{G1}^1 g_1^1(\mu)\exp(i\varphi)\,,\ \ G_1^{-1}(\mu,\varphi)=\left[\frac{3}{4\pi}\right]^{1/2} B_{G1}^1 g_1^1(\mu)\exp(i\varphi)\,, \tag{15}$$

$$B_{G1}^1=\left[\,\|x(\mu)\|^2-2\right]^{1/2}\left[\int_{-1}^{1}(1-\mu^2)\left[\frac{d}{d\mu}x(\mu)\right]^2 d\mu\right]^{-1/2}\,,$$

$$g_1^1(\mu)=\left\{\frac{3}{2}\left[\,\|x(\mu)\|^2-2\right]\right\}^{-1/2}(1-\mu^2)^{1/2}\frac{d}{d\mu}x(\mu)\,.$$

Here the light scattering indicatrix $x(\mu)$ is selected so that it satisfies the normalization condition:

$$\int_{-1}^{1} x(\mu)\,d\mu=2\,. \tag{16}$$

As is seen, Eqs. (15) for the explicit expressions of G functions contain the requirements of differentiability and quadratic integrability imposed on the initial scattering indicatrix. Note that in addition to the mathematical properties such as, for example, orthogonality, normalization, differentiability, and integrability, the G functions are also endowed with the physical properties inherent in the real scattering indicatrix $x(\mu)$ of a disperse medium that characterize the microstructure, shape, and anisotropy of particles, the optical constants of the substance of particles and of immersion, etc. Therein lies the characteristic feature of G functions. The orthogonal G functions (15) form the finite-dimensional G space consisting of two subspaces: a one-dimensional subspace that corresponds to the subscript l = 0 and a three-dimensional one with the subscript l = 1. In such a functional space the real light scattering indicatrix has an especially simple form. Thus, applying the summation theorem (10) for each of the subspaces:

$$g_0(\mu_\theta)=4\pi\sum_m G_0^0(\mathbf{\Omega}')\,G_0^0(\mathbf{\Omega})=1\,,\ \ l=0\,;$$

$$g_1(\mu_\theta)=\frac{4\pi}{3}\sum_m G_1^1(\mathbf{\Omega}')\,G_1^{1*}(\mathbf{\Omega})=g_1^0(\mu)\,g_1^0(\mu')+\left[B_{G1}^1\right]^2 g_1^1(\mu)\,g_1^1(\mu')\,2\cos(\varphi-\varphi')\,,\ \ l=1\,, \tag{17}$$

where $(\vartheta,\varphi)\in\{\mathbf{\Omega}\}$, $(\vartheta',\varphi')\in\{\mathbf{\Omega}'\}$ it can be easily shown that

$$x(\mu)=\sum_{l=0}^{1}\sum_{m=-l}^{l}[A_l]^{-2}\,x_l G_l^{*m}(\mathbf{\Omega})\,G_l^m(\mathbf{\Omega}')=1+x_1 g_1(\mu)\,. \tag{18}$$

Equation (18) determines an arbitrary scattering indicatrix $x(\mu)$ of a real medium. In a particular case, for an idealized linear indicatrix $x(\mu)$ widely used in the approximate transport theory [2] calculations by Eq. (18) yield $x(\mu)$ = $1+x_1P_1(\mu)$, where x_1 is the first coefficient of the expansion of the indicatrix in the Legendre polynomials $P_l(\mu)$.

It should be noted that in the basis of G functions the characteristic surface differs substantially from a single sphere of directions in a three-dimensional subspace. For an arbitrary scattering indicatrix we deal with a curved space whose curvature is determined by the character of the function $x(\mu)$. This means that the orthonormal G functions, in contrast to spherical ones, operate on a nonspherical characteristic surface, where the algebraic vector is assigned by the components $\Omega^l_{Gk}(\mathbf{\Omega})$, $l = 0, 1$; $k = 1, 2, 3$:

$$\Omega^0_{G1}(\mathbf{\Omega}) = 1 ,$$

$$\Omega^1_{G1}(\mathbf{\Omega}) = [A_1]^{-1}\left[B^1_1\right]^{-1}\frac{1}{2}\left[-G^1_1(\mu,\varphi) + G^{-1}_1(\mu,\varphi)\right] , \quad B^1_1 = \left[\frac{1}{2}\right]^{1/2} ,$$

$$\Omega^1_{G2}(\mathbf{\Omega}) = [A_1]^{-1}\left[B^1_1\right]^{-1}\frac{1}{2i}\left[-G^1_1(\mu,\varphi) - G^{-1}_1(\mu,\varphi)\right] , \tag{19}$$

$$\Omega^1_{G3}(\mathbf{\Omega}) = [A_1]^{-1}G^0_1(\mu,\varphi) .$$

In a particular case of a linear scattering indicatrix the characteristic surface (19) is degenerated into a single sphere of directions with the components $\Omega^l_k(\Omega)$:

$$\Omega^0_1(\mathbf{\Omega}) = 1 ,$$

$$\Omega^1_1(\mathbf{\Omega}) = \sin\vartheta\cos\varphi = \left[\frac{2\pi}{3}\right]^{1/2}\left[-Y^1_1(\mathbf{\Omega}) + Y^{-1}_1(\mathbf{\Omega})\right] ,$$

$$\Omega^1_2(\mathbf{\Omega}) = \sin\vartheta\sin\varphi = -i\left[\frac{2\pi}{3}\right]^{1/2}\left[Y^1_1(\mathbf{\Omega}) + Y^{-1}_1(\mathbf{\Omega})\right] , \tag{20}$$

$$\Omega^1_3(\mathbf{\Omega}) = \cos\vartheta = \left[\frac{4\pi}{3}\right]^{1/2} Y^0_1(\mathbf{\Omega}) ,$$

determined in terms of the spherical functions $Y^m_l(\mathbf{\Omega})$. From this it follows that in the three-dimensional subspace the vector $\mathbf{\Omega}^1_G = (\Omega^1_{G1}, \Omega^1_{G2}, \Omega^1_{G3})$ does not coincide with the vector $\mathbf{\Omega}^1 = (\Omega^1_1, \Omega^1_2, \Omega^1_3)$.

From Eq. (20) it is seen that to represent the direction of the beam $\mathbf{\Omega}$ in the differential operator of the RTE it is necessary to use the spherical functions $Y^m_l(\mathbf{\Omega})$. The possibility of applying the G functions in the RTE is ensured due to its property of biorthogonality which the G functions are endowed with relative to spherical functions:

$$\iint\limits_{\Omega} G^{*m}_l(\mu,\varphi)\, Y^{m'}_{l'}(\mu,\varphi)\, d\mu d\varphi = \delta^{mm'}_{ll'}\, \|G^m_l Y^m_l\| . \tag{21}$$

Here, the following designation of the binorm is used:

$$\|G^m_l Y^m_l\| = \iint\limits_{\Omega} G^{*m}_l(\mu,\varphi)\, Y^m_l(\mu,\varphi)\, d\mu d\varphi . \tag{22}$$

Note that the system of special G functions introduced in this way is the simplest system of orthogonal functions the expansion into which leads to the separation of variables in the radiation transfer equation. Thanks to this its exclusive property, one succeeds in separating the angular and spatial variables and in transforming the RTE to a system of differential equations for spatially dependent functions.

Transformation of RTE to a System of Differential Equations. The system of G functions together with the spherical functions $Y^m_l(\mathbf{\Omega})$ forms a consistent set of basis functions that is used in what follows for further transformations of RTE. First, we will consider the operator of radiation attenuation in a medium; in RTE (1) this operator corresponds to the Bouguer term $\varepsilon I(\mathbf{r}, \mathbf{\Omega})$ which will be represented as a functional similar to the collision integral:

$$\varepsilon I(\mathbf{r}, \mathbf{\Omega}) \equiv \frac{\varepsilon}{4\pi} \int_{\Omega'} x_{\varepsilon}(\mathbf{\Omega}\cdot\mathbf{\Omega}')\, I(\mathbf{r}, \mathbf{\Omega}')\, d\Omega' . \tag{23}$$

Here, the light attenuation indicatrix $x_{\varepsilon}(\mathbf{\Omega}\cdot\mathbf{\Omega}')$ expressed in terms of the Dirac δ function $x_{\varepsilon}(\mathbf{\Omega}\cdot\mathbf{\Omega}') = 4\pi\delta(\mathbf{\Omega}\cdot\mathbf{\Omega}' - 1)$ has been introduced. With allowance for the properties of the δ function we will find the coefficients of the expansion of the attenuation indicatrix $x_{\varepsilon}(\mathbf{\Omega}\cdot\mathbf{\Omega}')$ in the G functions:

$$x_{\varepsilon 0} = \frac{1}{2} \int_{-1}^{1} 2\delta(\mu - 1)\, g_0(\mu)\, d\mu = g_0(\mu = 1) , \tag{24}$$

$$x_{\varepsilon 1} = \frac{3}{2} \int_{-1}^{1} 2\delta(\mu - 1)\, g_1(\mu)\, d\mu = g_1(\mu = 1) . \tag{25}$$

It is seen that the expansion coefficient $x_{\varepsilon 1} = g_1^0(\mu = 1)$ depends on the value of the indicatrix of scattering in the direction $\vartheta = 0^{\circ}$. We will represent the function of sources $B_1(\mathbf{r}, \mathbf{\Omega})$ by the series

$$B_1(\mathbf{r}, \mathbf{\Omega}) = \sum_{l=0}^{N} \sum_{m=-l}^{l} A_l I_l'^{m}(\mathbf{r})\, Y_l^m(\mathbf{\Omega}) \tag{26}$$

with the expansion coefficients expressed by the equation

$$I_l'^{m}(\mathbf{r}) = [A_1]^{-1} \iint_{\Omega} B_1(\mathbf{r}, \mathbf{\Omega})\, Y_l^{*m}(\mathbf{\Omega})\, d\Omega , \quad l = 0, 1 ; \quad -l \le m \le l . \tag{27}$$

The solution for the intensity of diffuse radiation $I(\mathbf{r}, \mathbf{\Omega})$ will be represented in the form of a series in G functions:

$$I(\mathbf{r}, \mathbf{\Omega}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_l I_{Gl}^m(\mathbf{r})\, G_l^m(\mathbf{\Omega}) , \tag{28}$$

whose expansion coefficients $I_{Gl}^m(\mathbf{r})$ in the G functions are unknown as yet. The substitution of the above-indicated expansions into the RTE for the diffuse intensity, scattering indicatrix, attenuation indicatrix, vector of direction, and for the function of sources and the subsequent integration of the RTE over Ω with account taken of the conditions of orthogonality (9) and biorthogonality (21) lead to the differential equation

$$\nabla \mathbf{J}^1(\mathbf{r}) + 3\varepsilon(1 - \Lambda)\, J_1^0(\mathbf{r}) = 3\varepsilon\Lambda J_1'^0(\mathbf{r}) , \tag{29}$$

that couples the spatially dependent characteristics of scattered radiation $J_1^0(\mathbf{r})$ and $\mathbf{J}^1(\mathbf{r})$ and of the sources $J_1'^0(\mathbf{r})$. In the derivation of Eq. (29) we pass from the complex-valued quantities $I_{Gl}^m(\mathbf{r})$ in the expansion of the intensity in the G functions to their real components $J_{Gk}^l(\mathbf{r})$:

$$J_1^0(\mathbf{r}) = J_{G1}^0(\mathbf{r}) = [A_0]^2\, I_{G0}^0(\mathbf{r}) ,$$

$$J_1^1(\mathbf{r}) = \| G_1^1 Y_1^1 \|\, J_{G1}^1(\mathbf{r}) , \quad J_{G1}^1(\mathbf{r}) = [A_1]^2 B_1^1 \left[-I_{G1}^1(\mathbf{r}) + I_{G1}^{-1}(\mathbf{r}) \right] , \tag{30}$$

$$J_2^1(\mathbf{r}) = \|G_1^1 Y_1^1\| \, J_{G2}^1(\mathbf{r}) \,, \quad J_{G2}^1(\mathbf{r}) = i\,[A_1]^2 B_1^1 \left[-I_{G1}^1(\mathbf{r}) - I_{G1}^{-1}(\mathbf{r})\right] ,$$

$$J_3^1(\mathbf{r}) = \|G_1^0 Y_1^0\| \, J_{G3}^1(\mathbf{r}) \,, \quad J_{G3}^1(\mathbf{r}) = [A_1]^2 I_{G1}^0(\mathbf{r}) \,.$$

Here, $J_1^0(\mathbf{r})$ denotes the arithmetic mean intensity of scattered radiation in the one-dimensional subspace, and $J_1^1(\mathbf{r})$, $J_2^1(\mathbf{r})$, and $J_3^1(\mathbf{r})$ are the components of the light vector in the three-dimensional subspace. The second differential equation that couples $J_1^0(\mathbf{r})$ and $\mathbf{J}^1(\mathbf{r})$ results from computation of the integral over Ω of the RTE after it has been multiplied by $\mathbf{\Omega}$ and the above-indicated expansions substituted. It has the form

$$\nabla J_1^0(\mathbf{r}) + \varepsilon\,(f_{\varepsilon 1} - \Lambda f_1)\,\mathbf{J}^1(\mathbf{r}) = \varepsilon \Lambda f_1 \mathbf{J}'^1(\mathbf{r}) \,, \quad f_1 = \frac{x_1}{3} \,, \quad f_{\varepsilon 1} = \frac{x_{\varepsilon 1}}{3} \,, \tag{31}$$

where $\mathbf{J}'^1(\mathbf{r})$ is the light vector for the function of sources. Thus, the integrodifferential equation of radiation transfer (1), after being integrated in the basis of the special G space, is transformed to a system of deferential equations (29) and (31) for spatially dependent functions $J_1^0(\mathbf{r})$, $\mathbf{J}^1(\mathbf{r}) = (J_1^1, J_2^1, J_3^1)$, $J_1'^0(\mathbf{r})$, $\mathbf{J}'^1(\mathbf{r})$.

For the intensity $I(\mathbf{r}, \Omega)$ of diffuse radiation, with relations (19) and (30) taken into account, the following identical expansions are valid:

$$I(\mathbf{r}, \mathbf{\Omega}) = \sum_{l=0}^{1} \sum_{m=-l}^{l} A_l I_{Gl}^m(\mathbf{r})\, G_l^m(\mathbf{\Omega}) = \sum_{l=0}^{1} \sum_{k=1}^{3} J_{Gk}^l(\mathbf{r})\, \Omega_{Gk}^l(\mathbf{\Omega}) \,, \tag{32}$$

where the right-hand side has the form of a scalar product in the basis of real orthogonal components $\Omega_{Gk}^l(\Omega)$. Thus, in order to solve the RTE for the intensity $I(\mathbf{r}, \Omega)$, it is required to find the real components of the four-dimensional algebraic vector $\mathbf{J}_G(\mathbf{r}) = (J_{G1}^0, J_{G1}^1, J_{G2}^1, J_{G3}^1)$ by solving the system of differential equations (29), (31).

From Eq. (31) we express $\mathbf{J}'^1(\mathbf{r})$ in terms of $J_0^1(\mathbf{r})$:

$$\mathbf{J}^1(\mathbf{r}) = -\frac{1}{\varepsilon\,(f_{\varepsilon 1} - \Lambda f_1)} \left[\nabla J_1^0(\mathbf{r}) - \varepsilon \Lambda f_1 \boldsymbol{J}'^1(\mathbf{r})\right] \tag{33}$$

and substitute it into Eq. (29). As a result, we obtain a second-order partial differential equation for the mean-spherical intensity $J_1^0(\mathbf{r})$:

$$\Delta J_1^0(\mathbf{r}) - D^{-1} 3\varepsilon\,(1 - \Lambda)\, J_1^0(\mathbf{r}) = \varepsilon \Lambda f_1 \nabla \boldsymbol{J}'^1(\mathbf{r}) - D^{-1} 3\varepsilon \Lambda J_1'^0(\mathbf{r}) \,. \tag{34}$$

Here $D = [\varepsilon(f_{\varepsilon 1} - \Lambda f_1)]^{-1}$ is the generalized characteristic of the medium that has the meaning of the radiation diffusion length under the conditions of multiple scattering. With the notation

$$k^2 = D^{-1} 3\varepsilon\,(1 - \Lambda) \,, \quad F(\mathbf{r}) = D^{-1} 3\varepsilon \Lambda J_1'^0(\mathbf{r}) - \varepsilon \Lambda f_1 \nabla \mathbf{J}'^1(\mathbf{r}) \tag{35}$$

Eq. (34) takes the form

$$\Delta J_1^0(\mathbf{r}) - k^2 J_1^0(\mathbf{r}) = -F(\mathbf{r}) \,. \tag{36}$$

We will formulate the boundary condition for $J_1^0(\mathbf{r})$. Boundary condition (2) for the diffuse intensity expresses the fact that the scattered radiation which would be incident on the boundary of the medium is absent. In integral form it looks like

$$\int\limits_{(\Omega\cdot\mathbf{n})>0} I(\mathbf{r}, \mathbf{\Omega})\,(\mathbf{\Omega}\cdot\mathbf{n})\, d\Omega = 0 \,, \quad \mathbf{r} = \mathbf{r}_\Gamma \,, \quad \mathbf{n} = \mathbf{n}_\Gamma \,. \tag{37}$$

We rearrange Eq. (37) by the scheme applied to the RTE and make use of Eq. (33) at $\mathbf{r} = \mathbf{r}_\Gamma$. As a result, Eq. (37) is reduced to a system of two equations:

$$J_1^0(\mathbf{r}) + \frac{4}{3}(\mathbf{n}\cdot\mathbf{J}^1(\mathbf{r})) = 0\,, \quad \mathbf{r} = \mathbf{r}_\Gamma\,, \quad \mathbf{n} = \mathbf{n}_\Gamma\,; \tag{38}$$

$$\mathbf{J}^1(\mathbf{r}) = -\frac{1}{\varepsilon\,(f_{\varepsilon 1} - \Lambda f_1)}\nabla J_1^0(\mathbf{r})\,, \quad \mathbf{r} = \mathbf{r}_\Gamma\,, \tag{39}$$

where $J_1^1(\mathbf{r}) = \left| G_1^1 Y_1^1 \right|_\Gamma J_{G1}^1(\mathbf{r})$; $J_2^1(\mathbf{r}) = \left| G_1^1 Y_1^1 \right|_\Gamma J_{G2}^1(\mathbf{r})$; $J_3^1(\mathbf{r}) = \left| G_1^0 Y_1^0 \right|_\Gamma J_{G3}^1(\mathbf{r})$; $\left| G_l^m Y_l^m \right|_\Gamma = \int_0^1 \int_0^{2\pi} G_l^{*m}(\mu, \varphi) Y_l^m(\mu, \varphi)\, d\mu d\varphi$ is the binorm of the hemisphere of directions $(\mathbf{\Omega n}) > 0$ at the boundary of the medium. Substitution of Eq. (39) into Eq. (38) leads to the boundary condition of the third kind;

$$\mathbf{n}\nabla J_1^0(\mathbf{r}) - \eta J_1^0(\mathbf{r}) = 0\,, \quad \mathbf{r} = \mathbf{r}_\Gamma\,, \quad \mathbf{n} = \mathbf{n}_\Gamma\,, \tag{40}$$

where $\eta = (3/4)D^{-1}$ is the generalized parameter of the optical properties of the medium depending on the kind of the scattering indicatrix. Thus, in the basis of the G functions, boundary condition (40) follows strictly from Eq. (2) and replaces the well-known Marshack's, Mark's, and Davidson's approximate formulations [3].

Equation (36) with boundary condition (40) determines the boundary-value problem of the theory of radiation transfer for the spherical-mean intensity $J_1^0(\mathbf{r})$. It admits an analytical solution which is easily written in the form of a double series in eigenfunctions; however, such a form of representation presupposes an analysis of its convergence. At the same time, the solution given below has a closed form. It is based on the Green function for a three-dimensional boundary-value problem.

Solution of the Boundary-Value Problem for a Second-Order Partial Differential Equation. We will use the method of separation of variables [9] and represent $J_1^0(\mathbf{r})$ as the product $X(x)Y(y)Z(z)$, and thus we will reduce the boundary-value problem in partial derivatives (36), (40) to a system of boundary-value problems for one-dimensional functions $X(x)$, $Y(y)$, and $Z(z)$. We will write the boundary-value problems for the functions $X(x)$, $Y(y)$, and $Z(z)$ for ordinary differential equations of the type of the modified Helmholtz equation:

$$\frac{d^2X(x)}{dx^2} - \lambda_x^2 X(x) = 0\,; \quad \frac{dX(x)}{dx} - \eta X(x) = 0\,, \quad x = 0\,; \quad \frac{dX(x)}{dx} + \eta X(x) = 0\,, \quad x = a\,; \tag{41}$$

$$\frac{d^2Y(y)}{dy^2} - \lambda_y^2 Y(y) = 0\,; \quad \frac{dY(y)}{dy} - \eta Y(y) = 0\,, \quad y = 0\,; \quad \frac{dY(y)}{dy} + \eta Y(y) = 0\,, \quad y = b\,; \tag{42}$$

$$\frac{d^2Z(z)}{dz^2} - \lambda_z^2 Z(z) = 0\,; \quad \frac{dZ(z)}{dz} - \eta Z(z) = 0\,, \quad z = 0\,; \quad \frac{dZ(z)}{dz} + \eta Z(z) = 0\,, \quad z = c\,. \tag{43}$$

Here λ_x^2, λ_y^2, $\lambda_z^2 = k^2 - \lambda_x^2 - \lambda_y^2$ are the separation coefficients. We will find the Green function $G(x, x')$ of the boundary value problem (41) by taking advantage of the method of variation of arbitrary constants. The general solution of Eq. (41) (the subscript at λ is omitted) will be written in the form

$$X(x) = C_1(x')\exp(\lambda x) + C_2(x')\exp(-\lambda x)\,, \tag{44}$$

where $C_1(x')$ and $C_2(x')$ are variable coefficients. Allowance for the conditions on the boundaries $x = 0$ and $x = a$, as well as of the conditions of the solution continuity and of the jump of the derivative at the point $x = x'$ leads to an inhomogeneous system of four algebraic equations:

$$
\begin{gathered}
C_1^0(x')(\lambda-\eta)+C_2^0(x')(-\lambda-\eta)=0\,,\\
C_1^a(x')\,[\lambda\exp(\lambda a)+\eta\exp(\lambda a)]+C_2^a(x')\,[-\lambda\exp(-\lambda a)+\eta\exp(-\lambda a)]=0\,,\\
C_1^0(x')\,[-\exp(\lambda x')]+C_2^0(x')\,[-\exp(-\lambda x')]+C_1^a(x')\exp(\lambda x')+C_2^a(x')\exp(-\lambda x')=0\,,\\
C_1^a(x')\,[-\lambda\exp(\lambda x')]+C_2^a(x')\,[\lambda\exp(-\lambda x')]+C_1^a(x')\,[\lambda\exp(\lambda x')]+C_2^a(x')\,[-\lambda\exp(-\lambda x')]=-1\,.
\end{gathered}
\tag{45}
$$

We will calculate the unknown coefficients $C_1^0(x')$, $C_2^0(x')$, $C_1^a(x')$, and $C_2^a(x')$ and substitute them into Eq. (44). As a result of a number of transformations we will obtain an expression for the Green function:

$$
G(x,x')=\frac{1}{\Delta}\begin{cases}G_0(x,x')\,, & 0<x\le x'\,;\\ G_a(x,x')\,, & x'\le x<a\,,\end{cases}
\tag{46}
$$

where $G_0(x,x') = -[(\lambda+\eta)^2\exp[2\lambda(x-x'+a)] + (\lambda^2-\eta^2)\exp(2\lambda x) + (\lambda^2-\eta^2)\exp[2\lambda(a-x')] + (\lambda-\eta)^2]\times\exp[-\lambda(x-x'+a)]$; $G_a(x,x') = -\{(\lambda-\eta)^2\exp[-\lambda(a-x')] + (\lambda^2-\eta^2)\exp[-\lambda(a-3x')] + (\lambda^2-\eta^2)\times[\exp[-\lambda(2x-x'-a)] + (\lambda+\eta)^2\exp[-\lambda(2x-3x'-a)]\}\exp[\lambda(x-2x')]$, $\Delta = -2\lambda[(\lambda+\eta)^2\exp(2\lambda a) - (\lambda-\eta)^2]\times\exp(-\lambda a)$.

In addition to the arguments x, x', the Green function (46) contains three parameters λ, η, and a that determine the optical properties of the medium and the condition on the boundary. The values of λ satisfy the inequalities

$$
\lambda\ne 0\,,\quad \lambda\ne\frac{1}{a}\operatorname{arctanh}\left[-\frac{1}{2}\left(\frac{\lambda}{\eta}+\frac{\eta}{\lambda}\right)\right].
\tag{47}
$$

It can easily be seen that the Green function (46) is symmetrical relative to the change of the places of the arguments x and x'. In the simplest particular cases, from Eq. (46) the well-known expressions of the Green functions for the Dirichlet and Neumann problems follow.

We will invoke the principle of superposition of the influence of source points and write expressions for the one-dimensional functions $X(x)$, $Y(y)$, and $Z(z)$ in terms of the Green function (46) and $F(\mathbf{r}')$:

$$
\begin{gathered}
X(x)=\int_x^a G_0(x,x')\,dx'+\int_0^x G_a(x,x')\,dx'\,,\\
Y(y)=\int_y^b G_0(y,y')\,dy'+\int_0^y G_b(y,y')\,dy'\,,\\
Z(z)=\int_z^c F(z')\,G_0(z,z')\,dz'+\int_0^z F(z')\,G_c(z,z')\,dz'\,.
\end{gathered}
\tag{48}
$$

Then the solution of the boundary-value problem (36), (40) for the spherical-mean intensity $J_1^0(\mathbf{r})$ is expressed as a product

$$
J_1^0(\mathbf{r})=X(x)\,Y(y)\,Z(z)\,.
\tag{49}
$$

Analytical Solution of the Radiation Transfer Equation. The components $J_1^1(\mathbf{r})$, $J_2^1(\mathbf{r})$, and $J_3^1(\mathbf{r})$ of the three-dimensional vector $\mathbf{J}^1(\mathbf{r})$ will be found, according to (33), by differentiation of the function $J_1^0(\mathbf{r})$. As a result we have

$$J_1^1(\mathbf{r}) = -\frac{1}{\varepsilon(f_{\varepsilon 1} - \Lambda f_1)} \frac{\partial}{\partial x} X(x)\, Y(y)\, Z(z)\,, \tag{50}$$

$$J_2^1(\mathbf{r}) = -\frac{1}{\varepsilon(f_{\varepsilon 1} - \Lambda f_1)} X(x) \frac{\partial}{\partial y} Y(y)\, Z(z)\,, \tag{51}$$

$$J_3^1(\mathbf{r}) = -\frac{1}{\varepsilon(f_{\varepsilon 1} - \Lambda f_1)} \left[X(x)\, Y(y) \frac{\partial}{\partial z} Z(z) - \varepsilon \Lambda f_1 \mathbf{J}'(\mathbf{r}) \right]. \tag{52}$$

The solutions (49)–(52) obtained satisfy the system of differential equations (29), (31). With allowance for (30), (49)–(52), we will write expressions for the spatially dependent real components of scattered radiation $J_{G1}^0(\mathbf{r})$, $J_{G1}^1(\mathbf{r})$, $J_{G2}^1(\mathbf{r})$, and $J_{G3}^1(\mathbf{r})$ in the expansion of the intensity $I(\mathbf{r}, \mathbf{\Omega})$ in the G functions:

$$J_{G1}^0(\mathbf{r}) = J_1^0(\mathbf{r})\,, \quad J_{G1}^1(\mathbf{r}) = \|G_1^1 Y_1^1\|^{-1} J_1^1(\mathbf{r})\,, \quad J_{G2}^1(\mathbf{r}) = \|G_1^1 Y_1^1\|^{-1} J_2^1(\mathbf{r})\,, \quad J_{G3}^1(\mathbf{r}) = \|G_1^0 Y_1^0\|^{-1} J_3^1(\mathbf{r})\,. \tag{53}$$

The substitution of Eq. (53) into (32) leads to the sought solution $I(\mathbf{r}, \mathbf{\Omega})$ of the boundary-value problem (1), (2) of the transport theory for a three-dimensional disperse medium with an arbitrary scattering indicatrix $x(\mu)$ in the form of the following expression:

$$I(\mathbf{r}, \mathbf{\Omega}) = J_{G1}^0(\mathbf{r})\, \Omega_{G1}^0(\mathbf{\Omega}) + J_{G1}^1(\mathbf{r})\, \Omega_{G1}^1(\mathbf{\Omega}) + J_{G2}^1(\mathbf{r})\, \Omega_{G2}^1(\mathbf{\Omega}) + J_{G3}^1(\mathbf{r})\, \Omega_{G3}^1(\mathbf{\Omega})\,. \tag{54}$$

The result obtained can be conveniently formulated as follows. The radiation transfer equation (1) with boundary condition (2) for a three-dimensional volume of a disperse medium with an arbitrary scattering indicatrix has an analytical solution for the intensity of diffuse radiation in the form of expansion (54) with coefficients (53) in the basis of G functions (19) of a finite-dimensional functional space.

Conclusions. An analytical solution of the radiation transfer equation for a three-dimensional volume of a disperse medium with an arbitrary scattering indicatrix has been found. The first presentation of the solution obtained was made at the International Symposium of the CIS countries "Atmospheric Radiation" in St. Petersburg in 2002 [10]. The earlier unknown system of special G functions, connected with the RTE kernel and endowed with mathematical and physical properties, has been constructed. The system of G functions for an arbitrary scattering indicatrix is the simplest system of orthogonal functions the application of which leads to separation of angular and spatial variables and transformation of the RTE to a system of partial differential equations for spatially dependent characteristics of scattered radiation. The Green function of the boundary-value problem has been found for the spherical-mean intensity that ensured the closed character of solution of the RTE for a three-dimensional disperse medium. For an arbitrary real scattering indicatrix an analytical solution of the RTE has been obtained that has the form of an expansion in the basis of special G functions in the finite-dimensional functional space.

Analytical solutions for real scattering indicatrices of natural and artificial disperse media can find application in testing approximate, asymptotic, and numerical methods of transport theory. It appears natural to generalize the method of G functions to the solution of the vector radiation transfer equation [11, 12].

NOTATION

a, b, c, length, width, and height of a parallelepiped; $B_1(\mathbf{r}, \Omega)$, function of sources; D, length of radiation diffusion under the conditions of multiple scattering $F(\mathbf{r})$, function of sources for a partial differential equation; πF_0, flux of incident radiation; $g_l(\mu)$, one-dimensional G function; $g_l^m(\mu)$, associated one-dimensional G function; $G_l^m(\mathbf{\Omega})$, two-dimensional G function; $G(x, x')$, the Green function of the boundary-value problem for the modified Helmholtz equation in the final interval $[0, a]$; $|G_l^m Y_l^m|$, binorm of orthogonal functions; $|G_l^m Y_l^m|_\Gamma$, binorm of orthogonal functions on the boundary of the medium; $I(\mathbf{r}, \Omega)$, diffuse radiation intensity; $I_l'^m(\mathbf{r})$, coefficients of expansion of the function of sources; $I_{Gl}^m(\mathbf{r})$, coefficients of expansion of the intensity in G functions; $I_l'^l(\mathbf{r})$, real components of the coefficients of expansion of the function of sources; $U_j(\mu)$, jth initial independent function; $|x(\mu)|^2$, squared norm of the scattering indicatrix; **n**, internal normal to the boundary of the medium; $J^l_{Gk}(\mathbf{r})$, real components of the coefficients of expansion; $J^0_1(\mathbf{r})$, spherical mean intensity; **r**, radius vector of a point in space; x', coordinate of a point source; $x(\mu)$, light scattering indicatrix; x_1, the first coefficient of expansion of the scattering indicatrix in G functions; $x_\varepsilon(\mu)$, indicatrix of light attenuation; $x_{\varepsilon 1}$, first coefficient expansion of the attenuation indicatrix in G functions; $X(x)$, $Y(y)$, and $Z(z)$, one-dimensional functions for representation of the spherical-mean intensity; εz, optical depth; η, parameter of the optical properties of a medium in the vicinity of the boundary; θ, angle of scattering; ϑ, φ, polar and azimuthal angles of a spherical coordinate system; λ_x^2, λ_y^2, and λ_z^2, coefficients of separation; $\Psi_l(\mu)$, auxiliary function; $\mathbf{\Omega}_0$, vector of direction of the initial light beam; $\mathbf{\Omega}'$, vector of direction of the beam incident on an elementary volume of the medium; $\mathbf{\Omega}$, unit vector of the direction of scattered radiation; $\Omega^l_{Gk}(\mathbf{\Omega})$, components of basis vector $\mathbf{\Omega}_G{}'(\mathbf{\Omega})$ in a three-dimensional subspace; $\mathbf{\Omega}^1_G$ $(\mathbf{\Omega})$, radius vector of the characteristic surface in a three-dimensional subspace. Subscripts: Γ, boundary; ε, refers to attenuation.

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